B.sc Part II (Honrs)
Paper IV.

Exact differential Equations: — A differential equation is said to be exact if it can be derived from its primitive by direct differentiation without taking recourse to transformation such as elimination etc.

Example — The differential equation $xdy + ydx = 0$ is an exact differential equation as it is derived by direct differentiation for its solution the function
$$xy = c$$

Theorem: — The necessary and sufficient condition that the differential equation
$$Mdx + Ndy = 0$$
where M and N are functions of x and y is exact is that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Proof: — We shall first of all prove that the condition is necessary

i.e. We suppose that $Mdx + Ndy = 0$ is exact and we shall prove that
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Since $Mdx + Ndy = 0$ is given to be exact, therefore $Mdx + Ndy$ is an exact differential of some function say $u = f(x, y)$

i.e. $Mdx + Ndy = du$ ——— (1)

But we know from differential calculus that

$$du = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \quad \text{———} (2)$$

Therefore from (1) and (2), we get

$$Mdx + Ndy = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

It follows therefore

$$\frac{\partial \psi}{\partial x} = M \quad \text{————} (3)$$

and $\frac{\partial \psi}{\partial y} = N$ ———— (4)

Differentiating (3) Partially with respect to y and (4) partially w.r.t x, we get.

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Assuming that $\frac{\partial^2 \psi}{\partial y \partial x} \neq \frac{\partial^2 \psi}{\partial x \partial y}$ we obtain $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus we prove that the condition is ~~sufficient~~. necessary.

Now, we prove that the condition is sufficient.

That is given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we shall prove that

$Mdx + Ndy = d\psi$

$$\text{let } \int M dx = \phi \quad \therefore M = \frac{\partial \phi}{\partial x}$$

Again $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$$

$N = \frac{\partial \phi}{\partial y} + \psi(y)$ where $\psi(y)$ is a function of y.

Thus we have

$$Mdx + Ndy = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \psi(y) dy$$

$$= d(\phi) + \psi(y) dy$$

$$= d\{\phi + f(y)\} \text{ where } df(y) = \psi(y)$$

Now letting $\phi + f(y) = u$ where u is a function of x and y, we get $Mdx + Ndy = du$

Hence the result.

Example: 1. Solve $(2x - y + 1) dx + (2y - x - 1) dy = 0$

Solution:- Here, we have $M = 2x - y + 1$

$N = 2y - x - 1$ so that

$$\frac{\partial M}{\partial y} = -1 \text{ and } \frac{\partial N}{\partial x} = -1 \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact.

From the equation we have

$$2x dx - (y dx + x dy) + dx + 2y dy - dy = 0$$

$$\Rightarrow 2x dx - d(xy) + dx + 2y dy - dy = 0$$

Hence integrating, we get

$$2 \cdot \frac{x^2}{2} - xy + x + 2 \cdot \frac{y^2}{2} - y = C$$

i.e. $x^2 - xy + x + y^2 - y = c$

which is the required ~~solut~~ solution.

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